Some Thoughts on Teaching of p-Values

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Computational Science Hub

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z-Test for Population Mean μ

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- if H_0 is true, then

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \sim N(0, 1)$$

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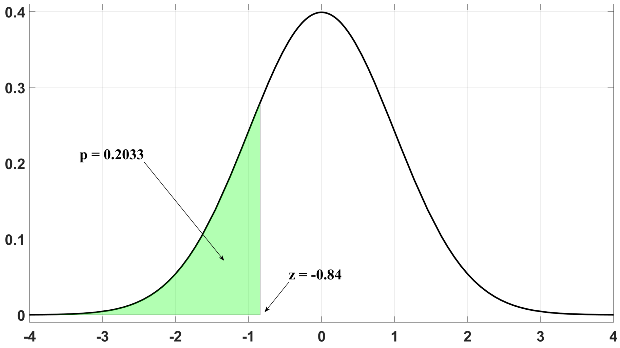
- if H_0 is true, then

$$Z = \frac{\bar{X} - \mu_0}{\sigma / \sqrt{n}} \sim N(0, 1)$$

- for a test of H_0 against $H_1 : \mu < \mu_0$ with observation z, the p-value is

$$p = \Pr(Z \leq z) = \Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}s^2} \, ds$$

Graphical Illustration



- the p-value is the area under the curve from $-\infty$ to z

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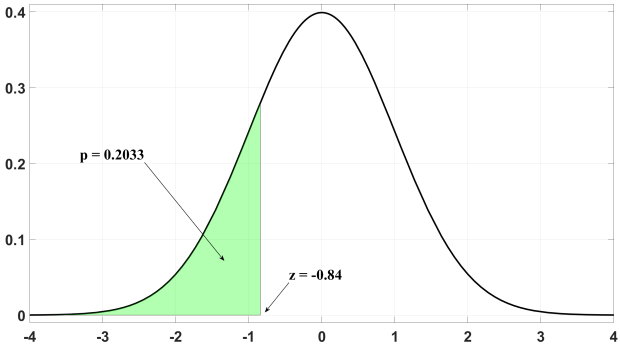

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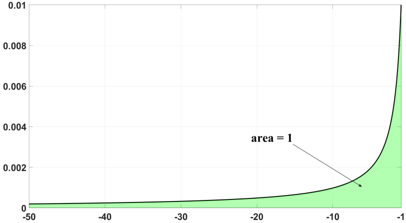
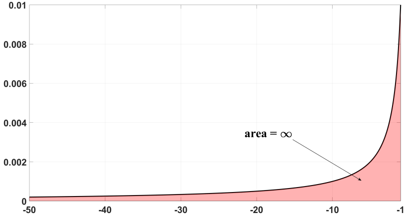
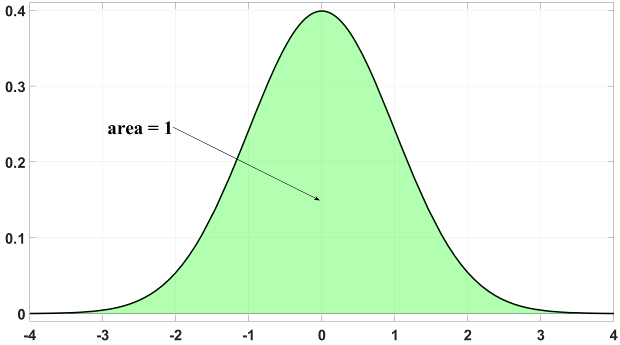
Improper Integrals are Non-Intuitive





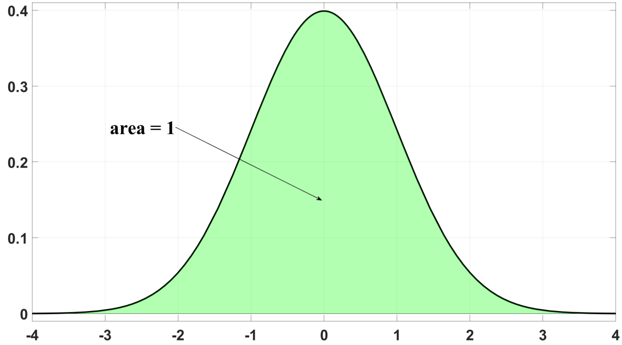
- the area under the curve from from $-\infty$ to -1

Area from $-\infty$ to ∞



- the area under the curve from $-\infty$ to ∞ is 1

Area from $-\infty$ to ∞



- the area under the curve from $-\infty$ to ∞ is 1

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}s^2} \, ds = 1$$

Improper Integrals are Non-Intuitive

- example of non-intuitivity

$$\int_{-z}^{z} s\,ds = 0 \text{ for any } z, \quad \text{but} \quad \int_{-\infty}^{\infty} s\,ds \text{ is not defined}$$

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- definitions of improper integrals

$$\int_{-\infty}^{z} f(s)\,ds = \lim_{a\to-\infty} \int_{a}^{z} f(s)\,ds$$

$$\int_{-\infty}^{\infty} f(s)\,ds = \lim_{a\to-\infty} \int_{a}^{0} f(s)\,ds + \lim_{b\to\infty} \int_{0}^{b} f(s)\,ds$$

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Distribution of the p-Value

- under H_0, the p-value is uniformly distributed with probability density function (pdf)

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$$f_1(p) = \frac{\phi\left(\Phi^{-1}(p) - \frac{\mu_1 - \mu_0}{\sigma/\sqrt{n}}\right)}{\phi(\Phi^{-1}(p))} \quad \text{with} \quad \phi(s) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}s^2}$$

Simulation Study

- $X \sim \mathcal{N}(\mu, 25)$, $H_0 : \mu = 325$, $H_1 : \mu = 322$, $n = 4$

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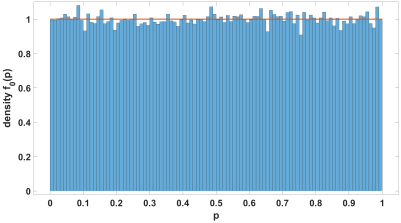

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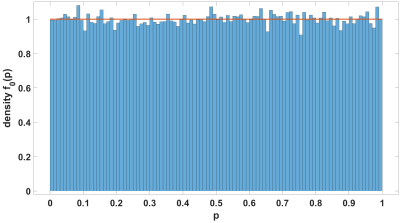

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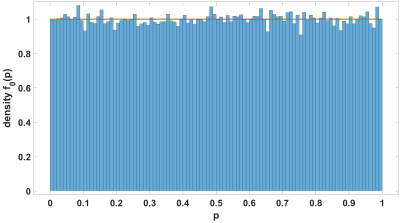

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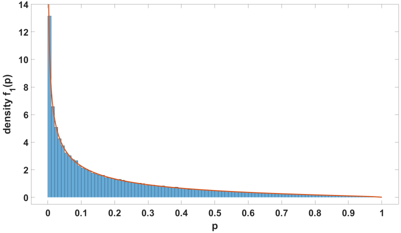

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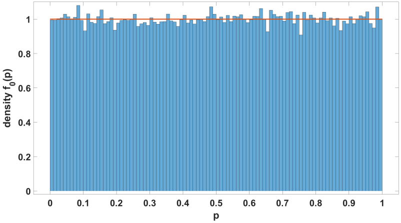

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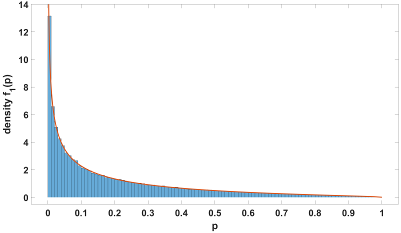

$$f_1(p) = \frac{\phi\left(\Phi^{-1}(p) + \frac{6}{5}\right)}{\phi(\Phi^{-1}(p))}$$

Bayes Factor

- Bayes' theorem yields

$$\frac{Pr(H_0|p)}{Pr(H_1|p)} = B(p) \cdot \frac{Pr(H_0)}{Pr(H_1)}$$

L. Held, M. Ott, Annu. Rev. Stat. Appl. (2018)

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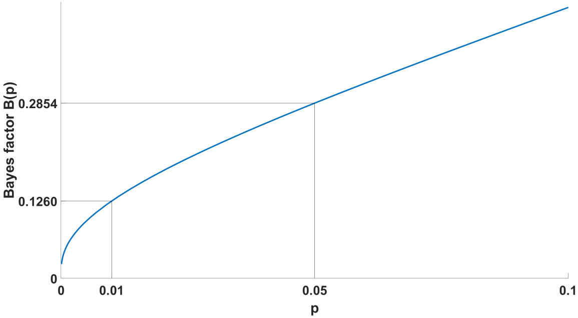
- the Bayes factor

$$B(p) = \frac{f_0(p)}{f_1(p)} = \frac{1}{f_1(p)}$$

measures how p increases or decreases the odds of H_0 to H_1

L. Held, M. Ott, Annu. Rev. Stat. Appl. (2018)

Example with $H_0 : \mu = 325$ and $H_1 : \mu = 322$



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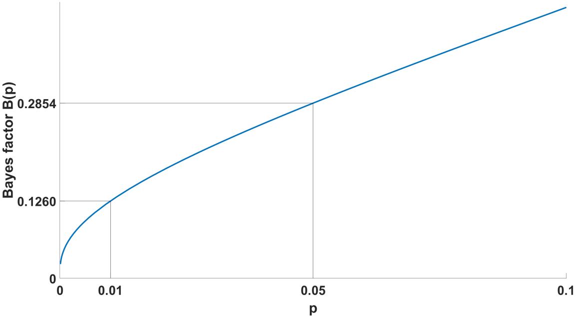

- suppose $Pr(H_0) = 0.9$ and $Pr(H_1) = 0.1$, then

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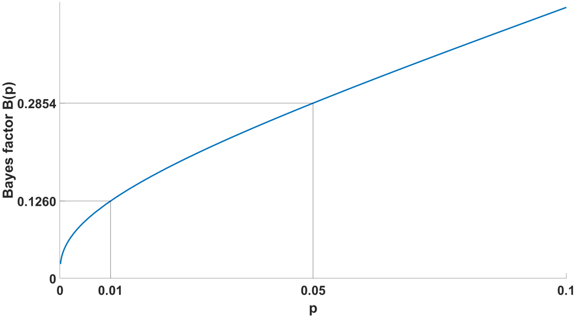

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  - $p = 0.05 \Rightarrow Pr(H_0|p) = 0.72$

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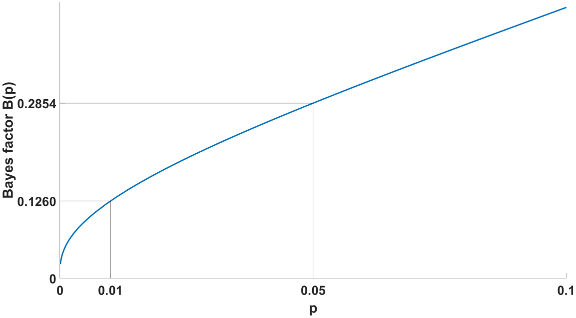

- suppose $Pr(H_0) = 0.9$ and $Pr(H_1) = 0.1$, then
  - $p = 0.05 \Rightarrow Pr(H_0|p) = 0.72$
  - $p = 0.01 \Rightarrow Pr(H_0|p) = 0.53$

Simulation Study with $Pr(H_0) = 0.9$

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- $N_0 = 90000$ and $N_1 = 10000$

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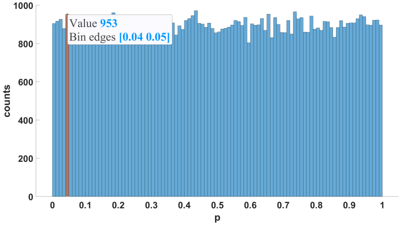

under H_0

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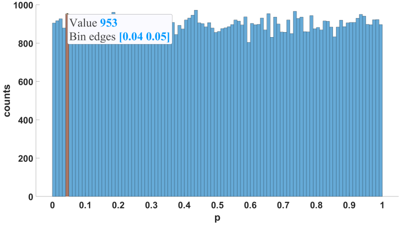

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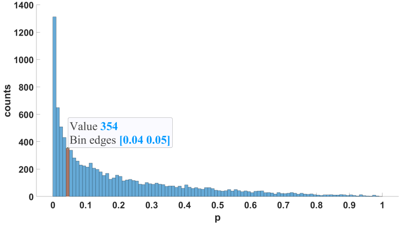

under H_1

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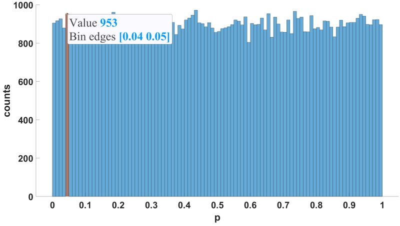

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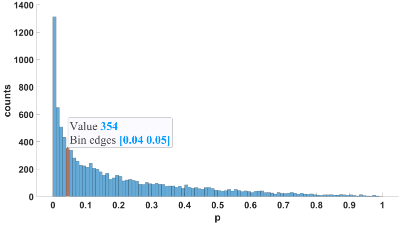

under H_1

- for about 72.9% of the tests with $p \in [0.04, 0.05]$, H_0 is true

$$\frac{953}{953 + 354} = 0.729 \approx Pr(H_0 | 0.05)$$

Bachelor Biology at ETH Zurich

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Mathematik (6 SWS)
Physik
Chemie
Praktikum Chemie

2. Semester

Grundlagen der Biologie 2: Zellen
Mathematik (5 SWS)
Physik
Chemie
Statistik (3 SWS)
Praktikum Biologie

3. Semester

Grundlagen der Biologie 3:
Multizellularität
Bioanalytik
Chemie
Informatik (4 SWS)
Statistik (4 SWS)
Praktikum Chemie

4. Semester

Genetik, Genomik
Biochemie
Systembiologie
Molekulare Mechanismen von
Gesundheit und Krankheit
Praktikum Biologie
Praktikum Bioinformatik (2 SWS)

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Praktikum Bioinformatik (2SWS)

- but what if statistics is not mandatory ?